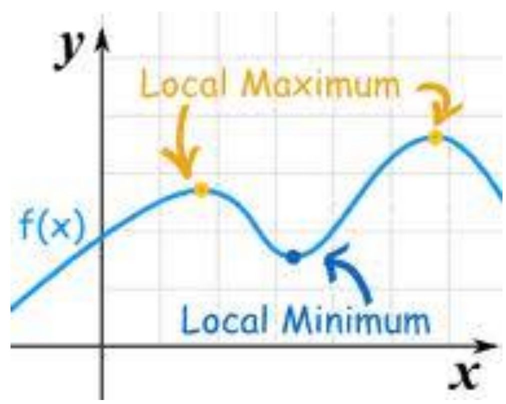
Introduction to **Finding the Max and Min values** of a function… and **Optimization**

In our everyday world around us it is not uncommon to want to either maximize or minimize certain quantities.

An engineer may want to maximize the speed of a new computer, or minimize the vibration in an appliance. A business may want to maximize profit and market share or minimize waste. Students want to maximize learning or minimize the hours needed to study.

Lots of ***natural phenomena*** also involve maximization or minimization. The surface area of an animal may minimize heat loss. Some plants want to maximize sunlight exposure. Prey animals learn to run in a path that minimizes their chance of getting eaten.

If we are able to model these or other situations with a *function,* calculus is the tool that will help find the best conditions for obtaining that max or min value that is desired.



Let's imagine that you are blindfolded and walking along a **road** that goes up and down. *You're really walking on top of a* ***function****…shhh!)*. Imagine that you are searching for the highest point on the road. How would you be able to decide when they were at the top of the hill?

Right at the top of the hill you would feel the road was level for a short period.
At the top of the hill <u>the slope would be zero</u>!

Without even being able to see the road, you would know that you were at the top of the hill when you were standing on level ground. The same idea would apply if the road had a valley in it, and you were searching for the lowest piece of road.

At the lowest point of the valley, the <u>slope of the road would be zero</u>.

So if we are searching for the highest (or lowest) point on a function, we need to consider those places where the **function has slope zero.** This is the idea behind Maximum and minimum values of a function.

Finding Max and Mins

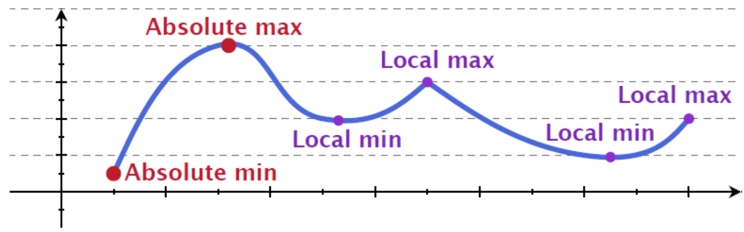
Critical points occur where:

a) f′(x) = 0

b) f′(x) = Does not exist

These are area to investigate

(but don't necessary mean max or mins)



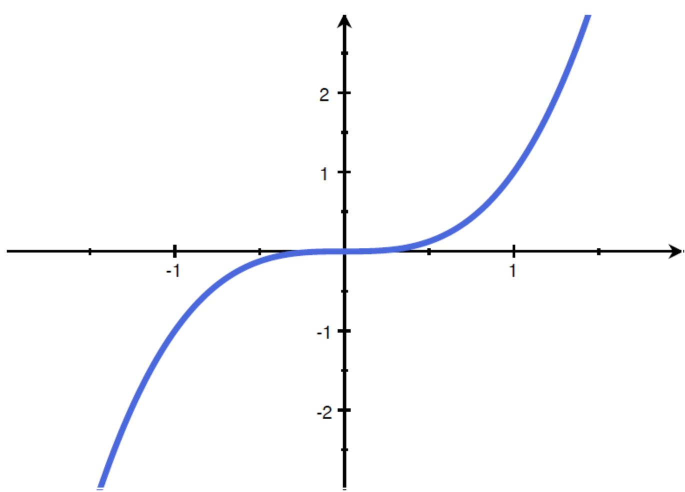
If f is continuous on a closed interval $[a, b]$, then there is a point in the interval where f is largest (**maximized**) and a point where f is smallest (**minimized**).

The maxima or minima will happen either

1. at an endpoint, or
2. at a **critical point**, a point c where $f'(c) = 0$ or $f'(c)$ is undefined.

Warning: Not all critical points are local minima or maxima:

Example: If $f(x) = x^3$, then $f'(x) = 3x^2$, and so $f'(0) = 0$:



Closed Interval Method for finding Absolute Max and Mins

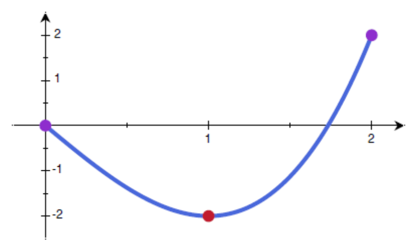
1. Calculate $f'(x)$.
2. Find where $f'(x)$ is 0 or undefined on $[a, b]$ (critical/singular points).
3. Evaluate $f(x)$ at the critical and singular points, and at endpoints. The largest (reps. smallest) value among these is the maximum (reps. minimum).

Example: Let $f(x) = x^3 - 3x$. What are the min/max values on the interval $[0, 2]$.

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

So $f'(x) = 0$ if $x = -1$ or $\boxed{1}$.

x	$f(x)$	
1	−2	critical points
0	0	end points
2	2	



The First derivative test (method)

Suppose

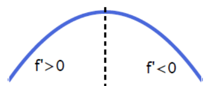
f is **continuous** on (a, b),

c is in (a, b) and is a **critical point** of $f(x)$, and

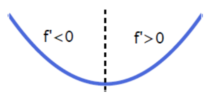
f is **differentiable** on (a, b) (except possibly at $x = c$)

Then the value $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from **positive** $\rightarrow$ **negative** at $x = c$, then $f(c)$ is a **local maximum**.



2. If $f'(x)$ changes from **negative** $\rightarrow$ **positive** at $x = c$, then $f(c)$ is a **local minimum**.



3. If $f'(x)$ doesn't change sign, then it's neither a min or a max.

EXAMPLE:

Example 2. Use the First Derivative Test to find the relative extrema of

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 12x + 1.$$

Give only the x-coordinate(s) of the extrema.

Solution. First find the critical numbers:

$$f'(x) = x^2 + x - 12.$$

So the critical numbers are

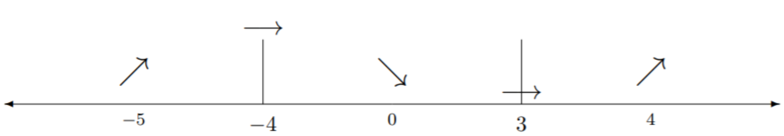
$$x^2 + x - 12 = 0$$

$$(x+4)(x-3) = 0$$

$$x + 4 = 0 \qquad x - 3 = 0$$

$$\boxed{x = -4} \qquad \boxed{x = 3}$$

Now **find the slope before and after these points** to determine if they are max or mins



$f'(-5) = (-5)^2 + (-5) - 12$ $\quad$ $f'(0) = (0)^2 + (0) - 12$ $\quad$ $f'(4) = (4)^2 + (4) - 12$

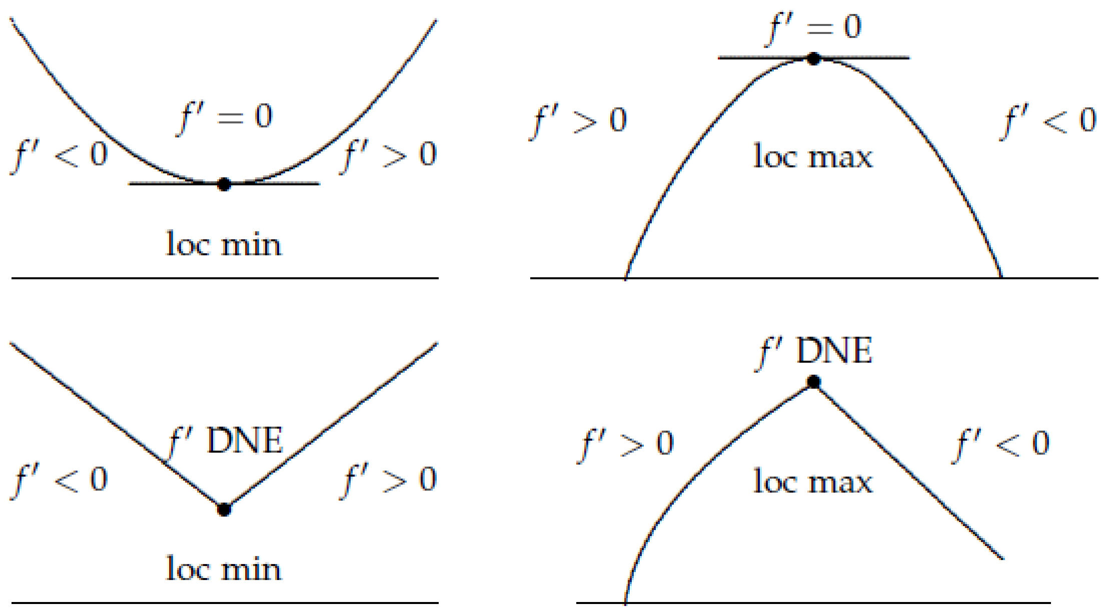
$f'(-5) = +8$ $\quad$ $f'(0) = -12$ $\quad$ $f'(4) = +8$

$\boxed{+}$ $\quad$ $\boxed{-}$ $\quad$ $\boxed{+}$

From the sign chart, we see that f has a relative maximum at $x = -4$ and a relative minimum at $x = 3$.

The First Derivative Test Examples:

We can find out if **Critical points** are *local max or mins* by testing out the slope of the graph before and after critical points.



Example#1:

Use the *first derivative* test to find any *local* max and min values for the functions below:

$$f(x) = x^3 - 3x^2 - 24x + 2$$

Example#2:

Use the *first derivative* test to find any *local* max and min values for the function below:

$$f(x) = (2x + 6)^4$$

Example#3:

Use the *first derivative* test to find any *local* max and min values for the function below:

$$f(x) = \ln(x^2 + 1)$$

The points c where $f'(c) = 0$ are so important, that make the definition:

critical number

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

We can rephrse Fermat's Theorem as "If f has a local min or max at c, then c is a critical number of f". And we now have an algorithm for maximizing or minimizing functions on closed intervals:

Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps (1) and (2) is the absolute maximum value; the smallest of these values is the absolute minimum value.

Find the Absolute max and min points in the interval [-2,4]
For the following function

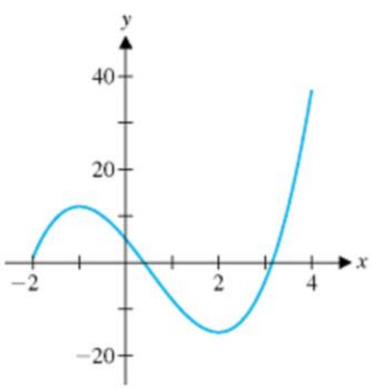


Figure 3.29
$y = 2x^3 - 3x^2 - 12x + 5.$

Workbook Problems:

Pg. 185 1-6 (look at examples)

Pg. 187 Ex1

Pg. 190 Ex3 #1-7

Pg. 192 Ex4 #1-6